Deformations of spaces of imbeddings

By ROBERT D. EDWARDS and ROBION C. KIRBY

Introduction

In [9], the second author proved that homeomorphisms of the torus T^n (the n-fold product of circles) are stable and used this fact, along with an immersion of T^n minus a point into R^n, to prove that the homeomorphism group of R^n (with the compact-open topology) is locally contractible. This paper generalizes the techniques of that paper and applies them to arbitrary manifolds. The main theorem of the paper says that if U is a subset of a manifold M containing a compact subset C in its interior, then given any two sufficiently close proper imbeddings $h_0, g_0: U \to M$ of U into M, there is a canonical proper isotopy $h_t: U \to M$, $t \in I$, connecting h_0 to an imbedding h_1 which agrees with g_0 on C. Furthermore, the isotopy is fixed on the complement of a compact neighborhood of C in U. As corollaries to the theorem we obtain the following.

COROLLARY 1.1. *The homeomorphism group $\mathcal{H}(M)$ of a compact manifold M is locally contractible.*

Remark. Since $\mathcal{H}(M)$ is a topological group, it follows rather easily that the point whose neighborhood is being contracted can be left fixed during the contraction. As it so happens, the contraction constructed in the proof has this additional property.

COROLLARY 1.2. *Let $h_t: C \to M$, $t \in I$, be a proper isotopy of a compact subset C of a manifold M such that h_t has a proper extension to a neighborhood U of C. Then h_t can be covered by an ambient isotopy of M, that is, there is an isotopy $H_t: M \to M$ such that $H_0 = 1_M$ and $h_t = H_t h_0$ for all t.*

If $h_t: M \to M$, $t \in I$, is an isotopy of M and B is a subset of M, then h_t is *supported* by B if $h_t \mid M - B = 1$ for all t.

COROLLARY 1.3. *Let $h_t: M \to M$, $t \in I$, be an isotopy of a compact manifold M and let $\{B_i \mid 1 \leqq i \leqq p\}$ be an open cover of M. Then h_t can be written as a composition of isotopies $h_t = h_{k,t} h_{k-1,t} \cdots h_{1,t} h_0$ where each isotopy $h_{j,t}: M \to M$ is an ambient isotopy which is supported by some member of $\{B_i\}$.*

Remark. It follows from the corollary that h_0 and h_1 are isotopic *via* arbitrarily small moves, that is, if $g_t: M \to M$, $t \in [0, k]$, is defined by $g_t =$

$h_{j,t-j+1}h_{j-1,1} \cdots h_{1,1}h_0$ for $t \in [j-1, j]$, then g_t is an isotopy by moves (each supported by some B_i) connecting $g_0 = h_0$ to $g_k = h_1$.

Section 7 of the paper shows how the above results can be extended to the relative case in which one considers a manifold pair (M, N), where N is a locally flat proper submanifold of M. Thus, close imbeddings of a subset U of M whose images of $U \cap N$ agree (as sets) are isotopic so that the image of $U \cap N$ stays invariant during the isotopy. Furthermore, if the imbeddings agree pointwise on $U \cap N$ then the isotopy is fixed on $U \cap N$. The above corollaries have the appropriate generalizations to the relative case. In particular, Corollary 1.2 can be substantially strengthened when C is a proper submanifold of M. If $h_t \colon N \to M$, $t \in I$, is a proper isotopy of a manifold N into M, then h_t is *locally flat* if for each $(x, t) \in N \times I$, there is a neighborhood $[t_0, t_1]$ of t in I and there are level preserving imbeddings $\alpha \colon B^n \times [t_0, t_1] \to N \times I$ and $\beta \colon B^n \times B^{m-n} \times [t_0, t_1] \to M \times I$ onto neighborhoods of (x, t) and $(h_t(x), t)$ respectively, such that the following diagram commutes.

$$\begin{array}{ccc} B^n \times 0 \times [t_0, t_1] & \subset & B^n \times B^{m-n} \times [t_0, t_1] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ N \times I & \xrightarrow[h(x,t)=(h_t(x),t)]{} & M \times I\,. \end{array}$$

If N is a locally flat proper submanifold of M, then this is equivalent to saying that for each $(x, t) \in N \times I$, the isotopy h_t extends to some neighborhood $U \times [t_0, t_1]$ of (x, t) in $M \times I$ in a level preserving fashion.

COROLLARY 1.4. (Isotopy extension theorem for topological manifolds.) *Let $h_t \colon N \to M$, $t \in I$, be a locally flat proper isotopy of a compact manifold N into a manifold M. Then h_t can be covered by an ambient isotopy of M, that is, there is an isotopy $H_t \colon M \to M$, $t \in I$, such that $H_0 = 1_M$ and $h_t = H_t h_0$ for all t.*

Corollary 1.1 has been proved independently by Cernavskii [3], using a convergent stretching-shrinking process on homeomorphisms instead of using torus homeomorphisms. Also, a version of Corollary 1.2 has been proved by Lees in [12], where he uses it to prove a topological version of a well-known immersion theorem from the differentiable and piecewise linear categories.

2. Notation and definitions

All manifolds are assumed to be metric, but are otherwise arbitrary. If U is a subset of a manifold M, a *proper* imbedding of U into M is an imbedding $h \colon U \to M$ such that $h^{-1}(\partial M) = U \cap \partial M$. An *isotopy* of U into M is a family of imbeddings $h_t\colon U \to M$, $t \in I$, such that the map $h\colon U \times I \to M$ defined by $h(x, t) = h_t(x)$ is continuous. An isotopy is *proper* if each imbedding in the isotopy is proper.

If C and U are subsets of M with $C \subset U$, let $I(U, C; M)$ denote the set of proper imbeddings of U into M which are the identity on C, and let $I(U; M)$ denote $I(U, \emptyset; M)$. Let $I(U, C; M)$ be provided with the compact-open topology. Thus a typical basic neighborhood of $h \in I(U, C; M)$ is of the form $N_h(K, \varepsilon) = \{g \in I(U, C; M) \mid d\big(g(x), h(x)\big) < \varepsilon$ for all $x \in K\}$, where K is a compact subset of U, $\varepsilon > 0$ and d is the metric on M.

Suppose X is a space with subsets A and B. *A deformation of A into B* is a map $\varphi\colon A \times I \to X$ such that $\varphi \mid A \times 0 = 1_A$ and $\varphi(A \times 1) \subset B$. If P is a subset of $I(U; M)$ and $\varphi\colon P \times I \to I(U; M)$ is a deformation of P, we may equivalently regard φ as a map $\varphi\colon P \times I \times U \to M$ such that for each $h \in P$ and $t \in I$, the map $\varphi(h, t, \_)\colon U \to M$ is a proper imbedding. Thus a deformation of P is simply a collection $\{h_t\colon U \to M, t \in I \mid h \in P\}$ of proper isotopies of U into M, continuously indexed by P, such that $h_0 = h$. All the deformations of the paper will be deformations of a neighborhood P of the inclusion $\eta\colon U \subset M$ in the space $I(U; M)$, and they will leave the inclusion fixed. If W is a subset of U, a deformation $\varphi\colon P \times I \to I(U; M)$ is *modulo W* if $\varphi(h, t) \mid W = h \mid W$ for all $h \in P$ and $t \in I$.

Suppose $\varphi\colon P \times I \to I(U; M)$ and $\psi\colon Q \times I \to I(U; M)$ are deformations of subsets of $I(U; M)$, and suppose that $\varphi(P \times 1) \subset Q$. Then the *composition of ψ with φ*, denoted by $\psi * \varphi$, is the deformation $\psi * \varphi\colon P \times I \to I(U; M)$ defined by

$$\psi * \varphi(h, t) = \begin{cases} \varphi(h, 2t) & \text{if } t \in [0, 1/2] \\ \psi\big(\varphi(h, 1), 2t - 1\big) & \text{if } t \in [1/2, 1] \, . \end{cases}$$

Let R^n be a euclidean n-space and let $B^n = [-1, 1]^n \subset R^n$. In general, let $aB^n = [-a, a]^n$ for $a > 0$, and let $[a, b]B^n = bB^n - \operatorname{int} aB^n$. We regard S^1 as the space obtained by identifying the endpoints of $[-4, 4]$ and we let $e\colon R^1 \to S^1$ denote the natural covering projection, that is, $e(x) = (x + 4) \bmod_8 -4$. Let T^n be the n-fold product of S^1. Then aB^n can be regarded as a subset of T^n for $a < 4$. Let $e^n\colon R^n \to T^n$ be the product covering projection and let $e^{k,n}\colon B^k \times R^n \to B^k \times T^n$ be the map $1_{B^k} \times e^n$. These maps will each be denoted by e when there is no possibility of confusion.

3. Propositions

Let D^n be the unit n-ball in R^n and let S^{n-1} be its boundary. We regard $S^{n-1} \times [-1, 1]$ as a subset of R^n by identifying (x, t) with $(1 + t/2)\cdot x$. The following proposition is a sort of ε-version of the generalized Schoenflies theorem [1]. It is a special case of a theorem of Huebsch and Morse [16, Th. 1.2] and has also been proved by Gauld [15].

PROPOSITION 3.1. (Canonical Schoenflies theorem.) *There exists an* $\varepsilon > 0$ *such that for any imbedding* $f: S^{n-1} \times [-1, 1] \to R^n$ *within* ε *of the identity,* $f \mid S^{n-1}$ *extends canonically to an imbedding* $\bar{f}: D^n \to R^n$. *The imbedding* $\bar{f}$ *is canonical in the sense that* $\bar{f}$ *depends continuously on* f *and if* $f = 1$, *then* $\bar{f} = 1$.

PROOF. It suffices to choose $\varepsilon > 0$ so that $f(S^{n-1} \times -1) \subset \operatorname{int} 3/4\, D^n$ and $f(S^{n-1} \times 0) \subset R^n - 3/4\, D^n$. The idea of the proof is to canonically define a map $\rho: f(S^{n-1} \times [-1, 0]) \to \operatorname{cl}(\operatorname{int} f(S^{n-1}))$ such that the only non-degenerate inverse set of ρ is $\rho^{-1}(0) = f(S^{n-1} \times -1)$ and such that $\rho \mid f(S^{n-1}) = 1$. Then ρ can be used to extend $f \mid S^{n-1}$ to $\bar{f}$ by defining $\bar{f}(0) = 0$ and $\bar{f} \mid D^n - 0 = \rho f \omega^{-1}$, where $\omega: S^{n-1} \times (-1, 0] \to D^n - 0$ is the homeomorphism defined by $\omega(x, t) = (1 + t)\cdot x$. If it happens that $\bar{f}$ is not the identity when f is, then let $\tau = \bar{1}: D^n \to D^n$ and replace $\bar{f}$ by $\bar{f}\tau^{-1}$.

The construction of ρ mimics the techniques used by Kister to show that an origin preserving imbedding of R^n into R^n can be deformed to be a homeomorphism [10, Th. 1]. These techniques are also written down in [11, Lem. 3]. Because of these references we omit the details here.

If M is a manifold, then a *collar* for ∂M is an imbedding $\sigma: \partial M \times [0, 1] \to M$ such that $\sigma \mid \partial M \times 0 = 1_{\partial M}$. The existence of such collars was proved in [2]. The following proposition is proved by an elementary application of one of the techniques of that paper. We will henceforth adopt the custom of identifying a collar with its image.

PROPOSITION 3.2. *Let* M *be a manifold with a collar* $\partial M \times [0, 1]$ *and let* C_0 *and* V_0 *be compact subsets of* ∂M *such that* $C_0 \subset \operatorname{int}_{\partial M} V_0$. *Let* U *be a subset of* M *such that* $V_0 \times [0, 1] \subset \operatorname{int} U$. *Then there is a neighborhood* P *of the inclusion* $\eta: U \subset M$ *in* $I(U, V_0; M)$ *and a deformation* $\varphi: P \times I \to I(U, V_0; M)$ *of* P *into* $I(U, V_0 \cup C_0 \times [0, 1/2]; M)$ *such that* φ *is modulo the complement of an arbitrarily small neighborhood of* $V_0 \times [0, 1]$.

PROOF. Let W be an arbitrary neighborhood of $V_0 \times [0, 1]$ in U and choose P so that $h \in P$ implies that $V_0 \times [0, 1] \subset h(W)$. Let $\lambda: V_0 \to [0, 1]$ be a map such that $\lambda(C_0) = 1$ and $\lambda(\operatorname{fr}_{\partial M} V_0) = 0$. For each $t \in [0, 1]$ let

$$W_t = \{(x, s) \in V_0 \times [0, 1] \mid 0 \leqq s \leqq t\lambda(x)\}$$

and define a homeomorphism $\gamma_t: W_t - \operatorname{int} W_{t/2} \to W_t$ by linearly stretching the fibers over the boundary points, that is, $\gamma_t(x, s) = (x, 2[s - t\lambda(x)/2])$ for each $(x, s) \in W_t - \operatorname{int} W_{t/2}$. Extend γ_t *via* the identity to a homeomorphism $\pi_t\colon \overline{M - W_{t/2}} \to M$. For each $h \in P$, define an isotopy $h_t\colon U \to M$, $t \in [0, 1]$, by

$$h_t = \begin{cases} \pi_t^{-1} h \pi_t & \text{on } U - W_{t/2} \\ 1 & \text{on } W_{t/2} . \end{cases}$$

Then $h_0 = h$, $h_1 \mid V_0 \cup C_0 \times [0, 1/2] = 1$ and $h_t \mid U - W = h \mid U - W$ for each t. This defines the desired deformation.

4. The main lemma

The following lemma is a special case of the main theorem, which in turn is proved by repeated applications of the lemma. The absolute case ($k = 0$) of the lemma was proved by the second author in [9, Th. 4]. The lemma will be generalized further in § 7, in order to handle the case where deformations leave fixed a submanifold of the given manifold. In § 8, we present an alternative proof of the lemma which makes use of a different technique to produce the intermediate homeomorphism of $B^k \times T^n$.

LEMMA 4.1. *There is a neighborhood Q of the inclusion $\eta\colon B^k \times 4B^n \subset B^k \times R^n$ in $I(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ and a deformation ψ of Q into*

$$I(B^k \times 4B^n, \partial B^k \times 4B^n \cup B^k \times B^n; B^k \times R^n)$$

modulo $\partial(B^k \times 4B^n)$ such that $\psi(\eta, t) = \eta$ for all t.

PROOF. Let C be the set $[1/2, 1]B^k \times 3B^n$. It is convenient to work with imbeddings which are the identity on C. This can be arranged by applying Proposition 3.2, which says that there exists a neighborhood Q_0 of η in $I(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ and a deformation

$$\psi_0\colon Q_0 \times I \to I(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$$

of Q_0 into $I(B^k \times 4B^n, \partial B^k \times 4B^n \cup C; B^k \times R^n)$ such that ψ_0 is modulo $\partial(B^k \times 4B^n)$. Note that $\psi_0(\eta, t) = \eta$ for all t.

The main construction of the lemma is as follows. Given an imbedding $h \in I(B^k \times 4B^n, \partial B^k \times 4B^n \cup C; B^k \times R^n)$ which is sufficiently close to the inclusion η, we construct a homeomorphism $g\colon B^k \times R^n \to B^k \times R^n$, continuously dependent upon h, such that $g \mid \partial B^k \times R^n \cup B^k \times (R^n - \operatorname{int} 3B^n) = 1$ and $g \mid B^k \times B^n = h \mid B^k \times B^n$. The deformation of the lemma is then defined by composing an isotopy of g with h. The homeomorphism g is produced by successively lifting maps as indicated in the diagram below.

$$
\begin{array}{ccc}
B^k \times R^n & \xrightarrow{g=h_5} & B^k \times R^n \\
\uparrow{\scriptstyle \gamma^{-1}} & & \uparrow{\scriptstyle \gamma^{-1}} \\
B^k \times R^n & \xrightarrow{h_4} & B^k \times R^n \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\
B^k \times T^n & \xrightarrow{h_3} & B^k \times T^n \\
\cup & & \cup \\
B^k \times T^n - 3D^k \times 3D^n & \xrightarrow{h_2|} & B^k \times T^n - D^k \times D^n \\
\cap & & \cap \\
B^k \times T^n - 2D^k \times 2D^n & \xrightarrow{h_2} & B^k \times T^n - D^k \times D^n \\
\cup & & \cup \\
B^k \times (T^n - 2D^n) & \xrightarrow{h_1} & B^k \times (T^n - D^n) \\
\downarrow{\scriptstyle \alpha|} & & \downarrow{\scriptstyle \alpha} \\
B^k \times 4B^n & \xrightarrow{h} & B^k \times R^n
\end{array}
$$

The neighborhood Q_1 of η which appears in the proof will always be understood to be a neighborhood of the inclusion map η in the space $I(B^k \times 4B^n, \partial B^k \times 4B^n \cup C; B^k \times R^n)$.

Let D^n, $2D^n$, $3D^n$, $4D^n$ be four concentric n-cells in $T^n - 2B^n$ such that $jD^n \subset \operatorname{int}(j+1)D^n$ for each j. Likewise let D^k, $2D^k$, $3D^k$, $4D^k$ be four concentric k-cells in int B^k such that $1/2B^k \subset D^k$ and $jD^k \subset \operatorname{int}(j+1)D^k$ for each j. As explained in [9, Prop. 3] or [13], there exists an immersion α_0: $T^n - D^n \to \operatorname{int} 3B^n$. By the generalized Schoenflies theorem [1], we can assume that $\alpha_0 \mid 2B^n$ is the identity. Let α denote the product immersion

$$1 \times \alpha_0 \colon B^k \times (T^n - D^n) \to B^k \times \operatorname{int} 3B^n .$$

If $h \in I(B^k \times 4B^n, \partial B^k \times 4B^n \cup C; B^k \times R^n)$ is close enough to η, then h can be covered in a natural way by an imbedding h_1: $B^k \times (T^n - 2D^n) \to B^k \times (T^n - D^n)$ (see diagram). This is done by defining h_1 to agree locally with $\alpha^{-1}h\alpha$. Thus, let $\{U_i \mid 1 \leqq i \leqq r\}$ be a finite cover of $T^n - \operatorname{int} 2D^n$ by open subsets of $T^n - D^n$ such that for any two members U_i, $U_{i'}$ which have non-empty intersection, $\alpha \mid (U_i \cup U_{i'})$ is an imbedding. Let $\{W_i \mid 1 \leqq i \leqq r\}$ be a cover of $B^k \times (T^n - 2D^n)$ by compact subsets of $B^k \times (T^n - D^n)$ such that $W_i \subset U_i$ for each i. If ε is chosen small enough and if $Q_1 = N_\eta(\alpha(\bigcup_{i=1}^r W_i), \varepsilon)$, then $h \in Q_1$ implies that $h\alpha(W_i) \subset \alpha(U_i)$ for each i. For such an h we can define the lifted map h_1: $B^k \times (T^n - 2D^n) \to B^k \times (T^n - D^n)$ by letting $h_1 \mid W_i = (\alpha \mid U_i)^{-1} h\alpha \mid W_i$ for each i. Then h_1 is an imbedding which lifts h and depends continuously on h, and is such that if h is the inclusion then so is h_1. Furthermore, $h_1 \mid (B^k - D^k) \times (T^n - 2D^n) = 1$.

From this latter property it follows that h_1 can be extended *via* the identity to an imbedding

$$h_2\colon B^k \times T^n - 2D^k \times 2D^n \to B^k \times T^n - D^k \times D^n .$$

Let $D^m = D^k \times D^n$. We can now apply the canonical Schoenflies theorem (Proposition 3.1) to extend $h_2 \mid B^k \times T^n - 3D^m$ to a homeomorphism of $B^k \times T^n$. For if Q_1 is sufficiently small then $h \in Q_1$ implies that $h_2 \mid (3\,1/2\,D^m - 2\,1/2\,D^m)$ is close to the identity and therefore $h_2 \mid \partial\, 3D^m\colon \partial\, 3D^m \to \operatorname{int} 4D^m$ extends to an imbedding $\bar{h}_2\colon 3D^m \to \operatorname{int} 4D^m$. Define a homeomorphism $h_3\colon B^k \times T^n \to B^k \times T^n$ by letting

$$h_3 \mid B^k \times T^n - 3D^m = h_2 \mid B^k \times T^n - 3D^m \text{ and } h_3 \mid 3D^m = \bar{h}_2 .$$

By the construction, h_3 depends continuously on h and if h is the inclusion, then h_3 is the identity.

Now if h_3 is sufficiently close to 1, then h_3 lifts in a natural way to a bounded homeomorphism $h_4\colon B^k \times R^n \to B^k \times R^n$ (where *bounded* means that the set $\{\| h_4(x) - x \| \mid x \in B^k \times R^n\}$ is bounded). We can define h_4 so that it locally agrees with $e^{-1}h_3e$, similar to the way that h_1 was defined. For if U is any subset of $B^k \times R^n$ of diameter < 4, then $e \mid U$ is an imbedding. For each $x \in B^k \times R^n$, let $h_4 \mid U_1(x) = \big(e \mid U_2(x)\big)^{-1}h_3e \mid U_1(x)$, where $U_\delta(x)$ denotes the open δ-neighborhood of x. Then h_4 depends continuously on h_3, $h_4 \mid \partial B^k \times R^n = 1$, and $h_4 = 1$ if $h_3 = 1$.

Let $\gamma\colon \operatorname{int} 3B^m \to R^m$ be a homeomorphism which is a radial expansion and which is the identity on $2B^m = 2B^k \times 2B^n$. Extend h_4 *via* the identity to a homeomorphism $h_4'\colon R^k \times R^n \to R^k \times R^n$ and define a homeomorphism $h_5\colon B^k \times R^n \to B^k \times R^n$ by

$$h_5 = \begin{cases} \gamma^{-1}h_4'\gamma & \text{on } B^k \times \operatorname{int} 3B^n \\ 1 & \text{on } B^k \times (R^n - \operatorname{int} 3B^n) . \end{cases}$$

The continuity of h_5 follows from the fact that h_4' is bounded. Now h_5 has the following properties:

(1) $h_5 \mid \partial B^k \times R^n \cup B^k \times (R^n - \operatorname{int} 3B^n) = 1$,

(2) $\alpha e\gamma h_5(x) = h\alpha e\gamma(x)$ for $x \in B^k \times 2B^n \cap h_5^{-1}(B^k \times 2B^n)$, and

(3) h_5 depends continuously on h, and if $h = \eta$, then $h_5 = 1$.

Property 3 implies that if Q_1 is small enough, then $h_5(B^k \times B^n) \subset B^k \times 2B^n$ whenever $h \in Q_1$. Thus, since $\alpha e\gamma \mid B^k \times 2B^n = 1$, property 2 implies that $h_5 \mid B^k \times B^n = h \mid B^k \times B^n$. Therefore h_5 is the desired map g mentioned at the beginning of the proof.

To complete the proof we show how to use $g = h_5$ to deform h to be the identity on $B^k \times B^n$. Extend g *via* the identity to a homeomorphism

$g\colon R^k \times R^n \to R^k \times R^n$ and define an isotopy $g_t\colon B^k \times R^n \to B^k \times R^n$, $t \in [0, 1]$, by using the Alexander trick on g, that is,

$$g_t(x) = \begin{cases} tg\left(\frac{1}{t}x\right) & \text{if } t > 0 \\ x & \text{if } t = 0\,. \end{cases}$$

Define a deformation

$$\psi_1\colon Q_1 \times I \to I(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$$

by $\psi_1(h, t) = g_t^{-1}h\colon B^k \times 4B^n \to B^k \times R^n$. Then ψ_1 deforms Q_1 into

$$I(B^k \times 4B^n, \partial B^k \times 4B^n \cup B^k \times B^n; B^k \times R^n)\,.$$

If Q_1 is small enough so that $h \in Q_1$ implies that $h(B^k \times \partial 4B^n) \cap B^k \times 3B^n = \emptyset$, then ψ_1 is modulo $\partial(B^k \times 4B^n)$. Note that $\psi_1(\eta, t) = \eta$ for all t. Finally, let Q be a neighborhood of η in $I(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$, $Q \subset Q_0$, such that $\psi_0(Q \times 1) \subset Q_1$, and let $\psi = \psi_1 * \psi_0 \mid Q \times I$. Then ψ is the desired deformation of the lemma.

5. The main theorem

The following theorem is the main result of the paper. Essentially it says that if U is a subset of a manifold M and if C is a compact subset in the interior of U, then any proper imbedding of U into M which is sufficiently close to the identity can be isotoped to an imbedding which is the identity on C, such that the isotopy itself is a continuous function of the imbedding. The theorem is proved by applying Lemma 4.1 to the handles of a carefully chosen handlebody cover of C.

THEOREM 5.1. (1) *Let M be a topological manifold and let C and U be subsets of M such that C is compact and U is a neighborhood of C. Given any neighborhood Q of the inclusion $\eta\colon U \subset M$ in $I(U; M)$, there is a neighborhood P of η in $I(U; M)$ and a deformation $\varphi\colon P \times I \to Q$ of P into $I(U, C; M)$. Furthermore, φ is modulo the complement of a compact neighborhood of C in U and $\varphi(\eta, t) = \eta$ for all $t \in I$.*

(2) *Suppose in addition to the above hypotheses that $\{D_1, D_2, \cdots, D_q\}$ is a finite collection of closed subsets of M, each with a neighborhood V_i in M. Then φ can be chosen so that the deformation $\varphi \mid [P \cap I(U, U \cap V_i; M)] \times I$ takes place in $I(U, U \cap D_i; M)$ for each i.*

Note. The deformation φ constructed in the proof has the additional property that if $h \in P$ is such that $h \mid U \cap \partial M = 1$, then $\varphi(h, t) \mid U \cap \partial M = 1$ for all t. Also, part 2 of the theorem holds for pairs (V_i, D_i) in ∂M where D_i is closed and V_i is a neighborhood of D_i in ∂M.

PROOF. The main portion of the proof is devoted to proving the following statement. *Given subsets C, D, U and V of M such that C is compact, D is closed, U is a neighborhood of C and V is a neighborhood of D, then there is a neighborhood P_η of η in $I(U, U \cap V; M)$ and a deformation $\varphi: P_\eta \times I \to I(U, U \cap D; M)$ of P_η into $I(U, U \cap (C \cup D); M)$ such that φ is modulo the complement of a compact neighborhood of C in U and $\varphi(\eta, t) = \eta$ for all t.* Part 1 of the theorem follows easily from the statement (letting $D = V = \emptyset$), for if Q is a given neighborhood of η in $I(U; M)$, then there is a neighborhood P of η in P_η such that $\varphi(P \times I) \subset Q$. Part 2 follows from the statement by means of an induction argument which is given at the end of the proof.

Henceforth it will be understood that all deformations of subsets of $I(U; M)$ fix the inclusion and are modulo the complement of a compact neighborhood of C in U.

The proof of the statement is divided into two cases.

Case 1. $\overline{C - D} \cap \partial M = \emptyset$. Let $\{(W_i, h_i) \mid 1 \leqq i \leqq r\}$ be a finite cover of $\overline{C - D}$ by coordinate neighborhoods which lie in U, where $h_i: W_i \to R^m$ is a homeomorphism. Express $\overline{C-D}$ as the union of r compact subsets $C_1, \cdots, C_r$ such that $C_i \subset W_i$, and let $D_i = D \cup \bigcup_{j \leqq i} C_j$ for $0 \leqq i \leqq r$. The proof of *case* 1 is by an induction argument on i. At the i^{th} step we assume that there exists a neighborhood P_i of $\eta: U \subset M$ in $I(U, U \cap V; M)$ and a deformation $\varphi_i: P_i \times I \to I(U, U \cap D; M)$ of P_i into $I(U, U \cap V_i; M)$, where V_i is some neighborhood of D_i. The induction starts trivially at $i = 0$ by taking $V_0 = V$, $P_0 = I(U, U \cap V; M)$ and φ_0 to be the identity deformation. We show how in general the inductive assumption can be extended to hold true for $i + 1$.

Identify W_{i+1} with R^m in order to simplify the notation. Then C_{i+1} is a compact subset of R^m and $V_i \cap R^m$ is a neighborhood in R^m of the closed subset $D_i \cap R^m$. There exists a finite cell complex pair (K, L) in R^m such that (K, L) has a handlebody decomposition with the following properties

(1) $D_i \cap C_{i+1} \subset L \subset \operatorname{int}(V_i \cap R^m)$,

(2) $C_{i+1} \subset K$,

(3) $\overline{K - L} \cap D_i = \emptyset$, and

(4) if A is a handle of $K - L$ and if k is the index of A, then there is an imbedding $\mu: B^k \times R^n \to R^m$, $m = k + n$, such that $\mu(B^k \times B^n) = A$ and $\mu(B^k \times R^n) \cap (D_i \cup L \cup \overline{K^k - A}) = \mu(\partial B^k \times B^n)$, where K^k denotes the union of all handles of K of index $\leqq k$.

The construction of (K, L) is standard, but for completeness we indicate how it is done. First, choose a compact neighborhood N of $C_{i+1} \cap D_i$ in

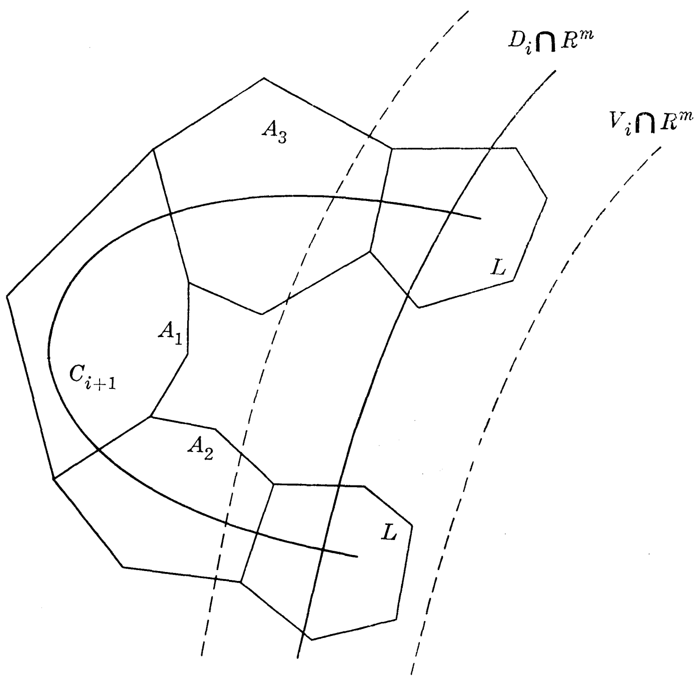

FIGURE 1

int $(V_i \cap R^m)$. Thus $C_{i+1} - N$ and $D_i - N$ have positive distance apart. Let T be a triangulation of R^m of mesh $< \varepsilon$, where $\varepsilon > 0$ is to be chosen small. Let K_0 and L_0 be the subcomplexes of T generated by the simplexes of T which intersect $C_{i+1} \cup N$ and N respectively. Then (K_0, L_0) is a simplicial pair in T. Let T' and T'' denote the first and second barycentric subdivisions of T and let $K = \bigcup \{\mathrm{st}\,(\hat{\sigma}, T'') \mid \sigma \in K_0\}$ and $L = \bigcup \{\mathrm{st}\,(\hat{\sigma}, T'') \mid \sigma \in L_0\}$, where $\hat{\sigma}$ denotes the barycenter of σ and $\mathrm{st}\,(\hat{\sigma}, T'')$ is the subcomplex of T'' generated by all the simplexes which intersect $\hat{\sigma}$. Each handle $\mathrm{st}\,(\hat{\sigma}, T'')$ is a polyhedral m-cell. Define its index to be dimension σ and let K^k denote the union of all the handles of index $\leqq k$. The two basic properties of the handlebody decomposition of K are

(a) if A and A' are two different handles of the same index, they are disjoint, and

(b) if A is a handle of index k, then the pair $(A, A \cap K^{k-1})$ is homeomorphic to the pair $(B^k \times B^n, \partial B^k \times B^n)$.

The fact that (K, L) has properties 1, 2, and 3 listed above follows immediately from the definition, assuming ε is chosen sufficiently small. Property 4 follows, for example, from property b above and the fact that ∂A is collared in R^m − int A. For a more detailed treatment of these facts, see [8, pp. 233 ff.].

Assume that $A_1, \cdots, A_j, \cdots, A_s$ are the handles of $K - L$ subscripted in order of increasing index. We proceed by induction on j to alter the imbeddings in $I(U, U \cap V_i; M)$ a step at a time in neighborhoods of the A_j's. For each j, $0 \leqq j \leqq s$, let $D'_j = D_i \cup L \cup \bigcup_{l \leqq j} A_l$ and assume inductively that for some neighborhood P'_j of η: $U \subset M$ in $I(U, U \cap V; M)$ there exists a deformation φ'_j: $P'_j \times I \to I(U, U \cap D; M)$ of P'_j into $I(U, U \cap V'_j; M)$ where V'_j is some neighborhood of D'_j in M. (If $j = 0$, the main inductive assumption gives precisely the information that is needed.) Consider A_{j+1} and the imbedding μ: $B^k \times R^n \to R^m$ given in property 4 above. By re-parametrizing the R^n coordinate if necessary, keeping B^n fixed, we can further assume that $\mu(\partial B^k \times 4B^n) \subset \operatorname{int} V'_j$.

According to Lemma 4.1 (replacing $B^k \times B^n$ by $B^k \times 2B^n$) there is a neighborhood Q of the inclusion η_0 in $I(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ and a deformation ψ of Q into $I(B^k \times 4B^n, \partial B^k \times 4B^n \cup B^k \times 2B^n; B^k \times R^n)$ modulo $\partial(B^k \times 4B^n)$ such that $\psi(\eta_0, t) = \eta_0$ for all t. Let Q' be a neighborhood of η in $I(U, U \cap V'_j; M)$ such that $h \in Q'$ implies that $h\mu(B^k \times 4B^n) \subset \mu(B^k \times R^n)$ and $\mu^{-1}h\mu \mid B^k \times 4B^n \in Q$. Then ψ can be used to define a deformation ψ': $Q' \times I \to I(U, U \cap D'_j; M)$ of Q' into $I(U, U \cap V'_{j+1}; M)$ as follows, where V'_{j+1} is a neighborhood of D'_{j+1} to be defined. If $h \in Q'$, define an isotopy h_t: $U \to M$, $t \in [0, 1]$, by

$$h_t = \begin{cases} h & \text{on } U - \mu(B^k \times 4B^n) \\ \mu\psi(\mu^{-1}h\mu, t)\mu^{-1} & \text{on } \mu(B^k \times 4B^n)\,. \end{cases}$$

Then $h_0 = h$ and $h_1 \in I(U, U \cap V'_{j+1}; M)$ where $V'_{j+1} = [V'_j \cup \mu(B^k \times 2B^n)] - \mu(B^k \times [2, 4]B^n)$. Let $\psi'(h, t) = h_t$. By the continuity of φ'_j there is a neighborhood P'_{j+1} of η in $I(U, U \cap V; M)$, $P'_{j+1} \subset P'_j$, such that $\varphi'_j(P'_{j+1} \times 1) \subset Q'$. Let

$$\varphi'_{j+1} = \psi' * (\varphi'_j \mid P'_{j+1} \times I): P'_{j+1} \times I \to I(U, U \cap D; M)\,.$$

Then φ'_{j+1} is the desired deformation, that is, φ'_{j+1} deforms P'_{j+1} into $I(U, U \cap V'_{j+1}; M)$.

At the completion of the subinduction argument on j, the main induction argument can be continued by taking $P_{i+1} = P'_s$, $V_{i+1} = V'_s$, and $\varphi_{i+1} = \varphi'_s$. This completes the proof of *case* 1.

Case 2. $\overline{C - D} \cap \partial M \neq \varnothing$. The idea of the proof of *case* 2 is to use a boundary collar for M and *case* 1 of the proof to initially deform the imbeddings to the identity on a neighborhood of $\overline{C - D} \cap \partial M$. The deformation can then be completed by applying *case* 1.

Let $\partial M \times [0, 1]$ be a boundary collar for M. Without loss of generality we can assume that $C = \overline{C - D}$ and that D is compact (since we can assume that U is compact and that $D \subset U$). Let C_0, D_0, U_0, and V_0 be subsets of ∂M and let $\varepsilon > 0$ be such that C_0 is compact, D_0 is closed, U_0 is a compact neighborhood of C_0, and V_0 is a neighborhood of D_0 and

$$C \cap (\partial M \times [0, 5\varepsilon]) \subset \operatorname{int}_{\partial M} C_0 \times [0, 5\varepsilon],\ U_0 \times [0, 5\varepsilon] \subset \operatorname{int} U\,,$$
$$D \cap (\partial M \times [0, 5\varepsilon]) \subset \operatorname{int}_{\partial M} D_0 \times [0, 5\varepsilon], \text{ and } V_0 \times [0, 5\varepsilon] \subset V\,.$$

Let C_1 be a compact neighborhood of C_0 in ∂M such that $C_1 \subset \operatorname{int}_{\partial M} U_0$.

The deformation φ produced for this case is the composition of three deformations φ_1, φ_2, and φ_3. The first deformation φ_1: $P_1 \times I \to I(U, U \cap D; M)$ deforms a neighborhood P_1 of η: $U \subset M$ in $I(U, U \cap V; M)$ into $I(U, U \cap (C_1 \cup V_1); M)$ modulo $U - U_0 \times [0, 5\varepsilon]$, where

$$V_1 = (V - \partial M \times [0, 5\varepsilon]) \cup D_0 \times [0, 5\varepsilon]\,,$$

which is a neighborhood of D. The second deformation φ_2, which is defined using Proposition 3.2, deforms a neighborhood P_2 of η in $I(U, U \cap (C_1 \cup V_1); M)$ into

$$I(U, U \cap (C_0 \times [0, 2\varepsilon] \cup V_1); M) \text{ modulo } U - U_0 \times [0, 5\varepsilon]\,.$$

At this stage the problem can be redefined so that *case* 1 of the proof applies, which leads to the definition of φ_3.

Definition of φ_1. It follows from *case* 1 that there is a neighborhood P_0 of the inclusion η_0: $U_0 \subset \partial M$ in $I(U_0, U_0 \cap V_0; \partial M)$ and a deformation φ_0: $P_0 \times I \to I(U_0, U_0 \cap D_0; \partial M)$ of P_0 into $I(U_0, U_0 \cap (C_1 \cup D_0); \partial M)$ modulo $\operatorname{fr}_{\partial M} U_0$. Let P_1 be a neighborhood of η: $U \subset M$ in $I(U, U \cap V; M)$ such that $h \in P_1$ implies that $h \mid U_0 \in P_0$. Given $h \in P_1$, use the deformation φ_0 to define a level preserving homeomorphism σ: $U_0 \times [0, 5\varepsilon] \to U_0 \times [0, 5\varepsilon]$ by letting

$$\sigma \mid U_0 \times t = (h \mid U_0)^{-1}\varphi_0\left(h \mid U_0, \frac{5\varepsilon - t}{5\varepsilon}\right)\,.$$

Then σ is the identity on $\operatorname{fr}_{\partial M} U_0 \times [0, 5\varepsilon] \cup U_0 \times 5\varepsilon$ and $\sigma \mid C_1 \times 0 = (h \mid U_0)^{-1} \mid C_1$. Extend σ to all of M *via* the identity. Then σ: $M \to M$ is isotopic to 1_M modulo $M - U_0 \times [0, 5\varepsilon]$ by the isotopy σ_t: $M \to M$, $t \in [0, 1]$, where σ_t is defined by

$$\sigma_t = \begin{cases} 1 & \text{on } M - \partial M \times [0, t] \\ \delta_t^{-1}\sigma\delta_t & \text{on } \partial M \times [0, t] \end{cases}$$

where δ_t: $\partial M \times [0, t] \to \partial M \times [1 - t, 1]$ is the homeomorphism which sends (x, s) to $(x, s + 1 - t)$. Let $\varphi_1(h, t) = h\sigma_t$.

Definition of φ_2. It follows from proposition 3.2 that there exists a neighborhood P_2 of η in $I(U, U \cap (C_1 \cup V_1); M)$ and a deformation $\varphi_2: P_2 \times I \to I(U, U \cap V_1; M)$ of P_2 into

$$I(U, U \cap (C_0 \times [0, 2\varepsilon] \cup V_1); M) \text{ modulo } U - U_0 \times [0, 5\varepsilon] .$$

One may take the $V_0 \times [0, 1]$ of the proposition to be $C_1 \times [0, 4\varepsilon]$ and may choose $U_0 \times [0, 5\varepsilon]$ to be the arbitrarily small neighborhood of $V_0 \times [0, 1]$. It follows from the proof of the proposition that an imbedding which is the identity on $U \cap V_1$ remains so during the deformation.

Definition of φ_3. Let $C_2 = C - \partial M \times [0, \varepsilon)$ and let $D_2 = D \cup (C \cap \partial M \times [0, \varepsilon])$. Then $C_2 \cup D_2 = C \cup D$ and $C_2 \cap \partial M = \varnothing$. Let $V_2 = C_0 \times [0, 2\varepsilon] \cup V_1$, which is a neighborhood of D_2. By *case* 1, there is a neighborhood P_3 of $\eta: U \subset M$ in $I(U, U \cap V_2; M)$ and a deformation $\varphi_3: P_3 \times I \to I(U, U \cap D_2; M)$ of P_3 into $I(U, U \cap (C_2 \cup D_2); M)$. This defines φ_3.

To conclude *case* 2, let $P_\eta \subset P_1$ be a neighborhood of $\eta: U \subset M$ in $I(U, U \cap V; M)$ such that $\varphi_1(P_\eta \times 1) \subset P_2$ and $\varphi_2(\varphi_1(P_\eta \times 1) \times 1) \subset P_3$ and define

$$\varphi = \varphi_3 * \varphi_2 * (\varphi_1 \mid P_\eta \times I): P_\eta \times I \to I(U, U \cap D; M) .$$

Then φ is the desired deformation.

We turn now to part 2 of the statement of the theorem. In order to be able to apply an induction argument we work with a generalized version of the statement given at the beginning of the proof, namely, *suppose C is a compact subset of M with a neighborhood U and suppose that $D, D_1, \cdots, D_q$ are closed subsets of M with neighborhoods $V, V_1, \cdots, V_q$, respectively. Then there is a neighborhood P_η of the inclusion $\eta: U \subset M$ in $I(U, U \cap V; M)$ and a deformation $\varphi: P_\eta \times I \to I(U, U \cap D; M)$ of P_η into $I(U, U \cap (C \cup D); M)$ such that*

$$\varphi([P \cap I(U, U \cap V_i; M)] \times I) \subset I(U, U \cap D_i; M)$$

for each i and such that φ is modulo the complement of a compact neighborhood of C in U and $\varphi(\eta, t) = \eta$ for all t. Part 2 of the theorem follows easily from this statement, letting $V = D = \varnothing$.

The proof is by induction on q. The case $q = 0$ is simply the original statement. In general, at the $(q+1)^{\text{st}}$ step, the statement is proved by making two applications of the statement with the value q, first to deform the imbeddings in a small neighborhood of $C \cap D_{q+1}$ and then to complete the deformation on the rest of C, away from D_{q+1}.

Choose compact sets C_0 and C_1 in int $(U \cap V_{q+1})$ such that C_0 is a neighborhood of $C \cap D_{q+1}$ and C_1 is a neighborhood of C_0. Let $U_1 = U \cap V_{q+1}$. Let

C_2 be a compact subset of C such that $C_2 \cap D_{q+1} = \emptyset$ and $C - C_0 \subset C_2$. Let $U_2 = U - D_{q+1}$, which is a neighborhood of C_2

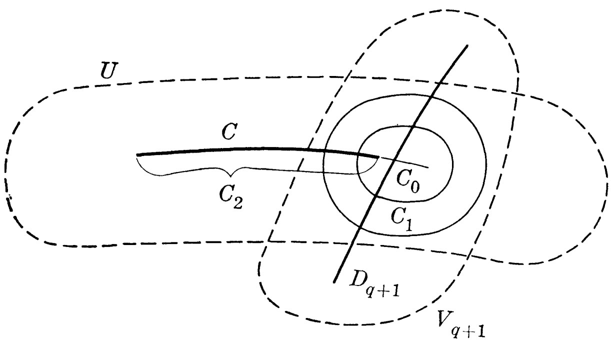

FIGURE 2

For each i, $1 \leqq i \leqq q$, let V_i' be a closed neighborhood of D_i such that $V_i' \subset \operatorname{int} V_i$. Likewise let V' be a closed neighborhood of D such that $V' \subset \operatorname{int} V$. Let $D_i' = V_i'$ and $D' = V'$.

By applying the induction hypothesis to the space $I(U_1, U_1 \cap V; M)$ and extending the deformation so obtained by the identity deformation on $U - U_1$, it follows that there exists a neighborhood P_1 of η in $I(U, U \cap V; M)$ and a deformation $\varphi_1 \colon P_1 \times I \to I(U, U \cap D'; M)$ of P_1 into $I(U, U \cap (D' \cup C_1); M)$ such that

$$\varphi_1([P_1 \cap I(U, U \cap V_i; M)] \times I) \subset I(U, U \cap D_i'; M)$$

for each i, $1 \leqq i \leqq q$. If we define D_{q+1}' to be D_{q+1}, then this last inclusion also holds for $i = q + 1$. Applying the induction hypothesis again, this time to the space $I(U_2, U_2 \cap (V' \cup C_1); M)$, and extending the deformation so obtained *via* the identity deformation on $U - U_2$, it follows that there exists a neighborhood P_2 of η in $I(U, U \cap (V' \cup C_1); M)$ and a deformation $\varphi_2 \colon P_2 \times I \to I(U, U \cap (D \cup C_0); M)$ of P_2 into $I(U, U \cap (D \cup C_0 \cup C_2); M)$ such that

$$\varphi_2([P_2 \cap I(U, U \cap V_i'; M)] \times I) \subset I(U, U \cap D_i; M)$$

for each i, $1 \leqq i \leqq q$. If we define V_{q+1}' to be D_{q+1}, then the inclusion also holds for $i = q + 1$. Let P_η be a sufficiently small neighborhood of η in P_1 such that $\varphi_1(P_\eta \times 1) \subset P_2$ and define φ to be the composition $\varphi = \varphi_2 * \varphi_1 \mid P_\eta \times I$. Then φ is the desired deformation of the statement. This completes the proof of the theorem.

6. Proofs of the corollaries

Let $\mathcal{H}(M)$ denote the group of homeomorphisms of a manifold M, provided with the compact-open topology. It is well known that $\mathcal{H}(M)$ is a topological group.

PROOF OF COROLLARY 1.1. Since $\mathcal{H}(M)$ is a topological group, it is enough to prove that $\mathcal{H}(M)$ is locally contractible at the identity. This follows immediately from part 1 of the theorem since $\mathcal{H}(M) = I(M; M)$ and $\{1_M\} = I(M, M; M)$.

Corollary 1.1 is in general no longer true in the case that M is not compact, since the compact-open topology is not fine enough to measure when two homeomorphisms are close enough to be isotopic. For example, let X be the space obtained from $S^1 \times R^1$ by deleting a countable number of small open discs, each centered at a point $(1, p)$, p an integer, and let M^2 be the unbounded 2-manifold obtained by sewing a copy of $T^2 - \operatorname{int} B^2$ to each of the boundary components of X. For any compact subset of M there is a homeomorphism which is the identity on the compact subset but is not isotopic to the identity since it may do some twisting of the manifold out near infinity. However, there is a certain class of manifolds for which Corollary 1.1 does hold true, namely those which are interiors of compact manifolds. In this case, $M = \operatorname{int} Q$ has an open collar $\partial Q \times (0, 1)$ induced by a collar for ∂Q in Q. The simplest example of such a manifold is R^n. The following corollary was proved for this special case in [9].

COROLLARY 6.1. *If M is a manifold which is homeomorphic to the interior of a compact manifold Q, then the homeomorphism group $\mathcal{H}(M)$ of M is locally contractible.*

PROOF. Let $\partial Q \times (0, 1)$ be an open collar for M, induced by a collar for Q. Let $C = M - \partial Q \times (0, 1/2)$, which is compact. By Theorem 5.1 there is a neighborhood P of the identity in $\mathcal{H}(M)$ and a deformation of P into $\mathcal{H}_1(M, C)$, where $\mathcal{H}_1(M, C)$ denotes the subgroup of homeomorphisms which are the identity on C. There is a natural deformation of $\mathcal{H}_1(M, C)$ into $\{1_M\}$ by making use of the open collar. By composing these deformations, it follows that P deforms into $\{1_M\}$ and therefore $\mathcal{H}(M)$ is locally contractible.

Another topology for the homeomorphism group $\mathcal{H}(M)$ which makes it into a topological group is the *majorant topology*. A typical basic neighborhood for a homeomorphism $h: M \to M$ in the majorant topology is of the form

$$N_h(\varepsilon(x)) = \{g \in \mathcal{H}(M) \mid d(g(x), h(x)) < \varepsilon(x) \text{ for all } x \in M\} ,$$

where d is the metric for M and $\varepsilon: M \to (0, \infty)$ is an arbitrary map. This topology is independent of the particular metric chosen for M. We denote

$\mathcal{H}(M)$ with this topology by $\mathcal{H}_m(M)$. Little can be said about deformations of subsets of $\mathcal{H}_m(M)$ that does not immediately follow from Theorem 5.1, since any two homeomorphisms which are connected by a path in $\mathcal{H}_m(M)$ must agree off a compact subset. However, by redefining the notion of deformation for this case we can say something about when nearby homeomorphisms are isotopic.

Let P and S be subsets of $\mathcal{H}_0(M)$. A *fine C^0 deformation* of P into S is a map $\varphi: P \times I \times M \to M$ such that for each $(h, t) \in P \times I$, the map $\varphi(h, t, \_): M \to M$ is a homeomorphism with $\varphi(h, 0, \_) = h$ and $\varphi(h, 1, \_) \in S$. If Q is a subset of $\mathcal{H}_0(M)$ and $\varphi(h, t, \_) \in Q$ for each h and t, then we say that the deformation *takes place* in Q.

Using this definition we have the following corollary.

COROLLARY 6.2. *Given any neighborhood Q of 1_M in $\mathcal{H}_m(M)$, there is a neighborhood P of 1_M in Q and there is a majorant deformation φ of P into $\{1_M\}$ such that φ takes place in Q.*

PROOF. The proof uses a standard technique that is often used to construct isotopies of a closed subset of a manifold by composing a countable number of isotopies, each of which is supported by some member of a locally finite compact cover of the closed set. Assume without loss of generality that M is connected. Let $\{(U_i, C_i) \mid 1 \leqq i < \infty\}$ be a countable collection of pairs of compact subsets of M such that for each i, U_i is a neighborhood of C_i, $M = \bigcup_{1 \leqq i} \operatorname{int} C_i$ and $U_i \cap U_j \neq \emptyset$ only if $|i - j| \leqq 1$. We can assume that Q is of the form $Q = N_1(\varepsilon(x))$, where $\varepsilon: M \to (0, \infty)$ is some map. Let $\varepsilon_i = \inf \varepsilon(U_i)$ for each i. It follows from part 2 of Theorem 5.1 (letting $U = U_{2i}$, $C = C_{2i}$, $V_1 = C_{2i-1} \cup C_{2i+1}$, and $D_1 = \overline{U_{2i} - C_{2i}}$) that there is a sequence $\{\delta_{2i}\}$ of positive numbers such that if P_{2i} is defined to be the neighborhood $N_\eta(U_{2i}, \delta_{2i})$ of $\eta: U_{2i} \subset M$ in $I(U_{2i}; M)$, then there is a deformation $\varphi_{2i}: P_{2i} \times I \to I(U_{2i}; M)$ of P_{2i} into $I(U_{2i}, C_{2i}; M)$ such that φ_{2i} deforms $P_{2i} \cap I\big(U_{2i}, U_{2i} \cap (C_{2i-1} \cup C_{2i+1}); M\big)$ into $\{\eta\}$, φ_{2i} takes place in $N_\eta(U_{2i}, \varepsilon_{2i})$, and φ_{2i} is modulo $\operatorname{fr}_M U_{2i}$. Likewise, there is a sequence $\{\delta_{2i-1}\}$ of positive numbers such that if P_{2i-1} is defined to be the neighborhood $N_\eta(U_{2i-1}, \delta_{2i-1})$ of $\eta: U_{2i-1} \subset M$ in $I(U_{2i-1}; M)$, then there is a deformation $\varphi_{2i-1}: P_{2i-1} \times I \to I(U_{2i-1}; M)$ of P_{2i-1} into $I(U_{2i-1}, C_{2i-1}; M)$ such that φ_{2i-1} takes place in

$$N_\eta\big(U_{2i-1}, \min\{\varepsilon_{2i-1}, \delta_{2i-2}, \delta_{2i}\}\big)$$

and φ_{2i-1} is modulo $\operatorname{fr}_M U_{2i-1}$. Let $\delta: M \to (0, \infty)$ be such that $\sup \delta(U_i) \leqq \delta_i$ for each i and let P be the $\delta(x)$-neighborhood of 1_M in $\mathcal{H}_m(M)$. Define a deformation $\varphi: P \to \mathcal{H}_m(M \times I)$ by

$$\varphi(h)_t = \begin{cases} \varphi_{2i-1}(h \mid U_{2i-1}, 2t) & \text{on } U_{2i-1} \text{ for } t \in [0, 1/2], \text{ and} \\ h & \text{on } M - \bigcup_{1 \leqq i} U_{2i-1} \end{cases}$$

$$\varphi(h)_t = \begin{cases} \varphi_{2i}(\varphi(h)_{1/2} \mid U_{2i}, 2t - 1) & \text{on } U_{2i} \text{ for } t \in [1/2, 1], \text{ and} \\ \varphi(h)_{1/2} & \text{on } M - \bigcup_{1 \leqq i} U_{2i} \end{cases}$$

Then φ is the desired deformation.

Remark. The technique used in the above proof can also be used to generalize Theorem 5.1 to the consideration of proper imbeddings of a neighborhood U of a closed subset C of a manifold M. In this case $I(U, C; M)$ is provided with the majorant topology and the deformations considered are majorant deformations, defined in a manner analogous to the above definition.

PROOF OF COROLLARY 1.2. Let h_t: $U \to M$, $t \in I$, denote the extended proper isotopy. Consider an arbitrary $t_0 \in I$. By Theorem 5.1, there is a neighborhood P of the inclusion η: $h_{t_0}(U) \subset M$ in $I(h_{t_0}(U); M)$ and a deformation φ: $P \times I \to I(h_{t_0}(U); M)$ of P into $I(h_{t_0}(U), h_{t_0}(C); M)$ such that φ is modulo $\mathrm{fr}_M\, h_{t_0}(U)$. Let $N(t_0)$ be a sufficiently small neighborhood of t_0 in I such that $h_t h_{t_0}^{-1} \in P$ for all $t \in N(t_0)$, and define a partial isotopy $H_{t_0,t}$: $M \to M$, $t \in N(t_0)$, by

$$H_{t_0,t} = \begin{cases} h_t h_{t_0}^{-1} \varphi^{-1}(h_t h_{t_0}^{-1}, 1) & \text{on } h_t(U) \\ 1 & \text{on } M - h_t(U)\ . \end{cases}$$

Then $h_t \mid C = H_{t_0,t} h_{t_0} \mid C$ for all $t \in N(t_0)$.

The proof now proceeds as in the piecewise linear case [7]; see also [8, p. 150]. Using the compactness of I, choose a partition $0 = t_0 < t_1 \cdots < t_n = 1$ of I and a collection of isotopies $H_{i,t}$: $M \to M$, $t \in [t_i, t_{i+1}]$, such that $h_t \mid C = H_{i,t} h_{t_i} \mid C$ for $t \in [t_i, t_{i+1}]$. Let $H_0 = 1_M$ and assume inductively that there is an ambient isotopy H_t: $M \to M$, $t \in [0, t_i]$, such that $h_t \mid C = H_t h_0 \mid C$ for $t \in [0, t_i]$. Extend H_t to the interval $[t_i, t_{i+1}]$ by letting $H_t = H_{i,t} H_{i,t_i}^{-1} H_{t_i}$ for $t \in [t_i, t_{i+1}]$. Then H_t: $M \to M$, $t \in [0, 1]$, is the desired covering isotopy for h_t.

Remark. Corollary 1.2 has various improvements, just as in the piecewise linear case [7, 8]. For example, if D is the track of C under h_t, that is, $D = \bigcup_{t \in I} h_t(C)$, then H_t may be chosen to be the identity off any given neighborhood of D. Also, if $h_t \mid U \cap \partial M = h_0 \mid U \cap \partial M$ for all t, then H_t can be chosen so that $H_t \mid \partial M = 1$.

PROOF OF COROLLARY 1.3. We first show that there is a neighborhood P of the identity in $\mathcal{H}(M)$ such that each $h \in P$ can be canonically written as a composition $h = h_p h_{p-1} \cdots h_1$ of p homeomorphisms such that h_i is supported by B_i for each i. This is proved by making p applications of part 2 of Theorem 5.1 (with the value $q = 1$; actually, all that we use is the statement given in the first paragraph of the proof of the theorem). First it is necessary to shrink the cover $\{B_i\}$ p times in order to provide some cushioning neighborhoods. For each i, $1 \leqq i \leqq p$, let $B_{i,0} = B_i$ and let $\{B_{i,j} \mid 1 \leqq j \leqq p\}$ be a collection of p compact subsets of M such that $B_{i,j} \subset \operatorname{int} B_{i,j-1}$, $1 \leqq j \leqq p$, and such that $M = \bigcup_{1 \leqq i \leqq p} B_{i,p}$. For each i, $1 \leqq i \leqq p$, Theorem 5.1 implies that there is a neighborhood P_i of 1_M in $I(M, \bigcup_{1 \leqq k \leqq i-1} B_{k,i-1}; M)$ and a deformation $\varphi_i\colon P_i \times I \to \mathcal{H}(M)$ of P_i into $I(M, \bigcup_{1 \leqq k \leqq i} B_{k,i}; M)$ such that φ_i is modulo $M - B_{i,0}$ and $\varphi_i(1_M, t) = 1_M$ for all t (we are applying the theorem with $C = B_{i,i}$, $U = B_{i,0}$, $D_1 = \bigcup_{1 \leqq k \leqq i-1} B_{k,i}$ and $V_1 = \bigcup_{1 \leqq k \leqq i-1} B_{k,i-1}$). Let P be a sufficiently small neighborhood of 1_M in $\mathcal{H}(M)$ such that each of the deformations $\varphi_i * \cdots * \varphi_1 \mid P \times I$ is defined. For $h \in P$, let $g_0 = h$ and let $g_i = \varphi_i * \cdots * \varphi_1(h, 1)$. Then $g_p = 1_M$ and therefore h $= (g_p^{-1} g_{p-1}) \cdots (g_2^{-1} g_1)(g_1^{-1} g_0)$ expresses h in the desired manner. Clearly this process depends continuously on h, so that if $h_t\colon M \to M$, $t \in I$, is an ambient isotopy such that $h_t \in P$ for all t, then h_t can be expressed as a composition of ambient isotopies $h_t = h_{p,t} h_{p-1,t} \cdots h_{1,t}\colon M \to M$ such that for each i, $h_{i,t}$ is supported by B_i.

Now let $h_t\colon M \to M$, $t \in I$, be an arbitrary isotopy. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of I such that for each j, $0 \leqq j < n$, and each $t \in [t_j, t_{j+1}]$, $h_t h_{t_j}^{-1} \in P$. For each j, define an ambient isotopy $g_{j,t}\colon M \to M$, $t \in I$, by

$$g_{j,t} = \begin{cases} 1 & \text{if } t < t_j \\ h_t h_{t_j}^{-1} & \text{if } t_j \leqq t \leqq t_{j+1} \\ h_{t_{j+1}} h_{t_j}^{-1} & \text{if } t_{j+1} < t. \end{cases}$$

Then $h_t h_0^{-1} = g_{n-1,t} g_{n-2,t} \cdots g_{0,t}$, and $g_{j,t} \in P$ for each j and t. By the above remarks, each isotopy $g_{j,t}$ can be expressed as a composition of isotopies with the desired properties, and therefore so can $h_t h_0^{-1}$.

Let N be the normal subgroup of $\mathcal{H}(M)$ generated by homeomorphisms which are supported by a proper ball in M, where a proper ball is defined to be a ball B such that $B \cap \partial M = \varnothing$ or $B \cap \partial M$ is an $(m-1)$-ball. It follows from the corollary that N is the subgroup of homeomorphisms which are isotopic to the identity, since one may choose to cover M by proper balls. Note that N is open by Corollary 1.1. If $\partial M = \varnothing$, it is known that N is the smallest normal subgroup of $\mathcal{H}(M)$ [4] (if $\partial M \neq \varnothing$, then this relation is no longer true, since the smallest normal subgroup is generated by homeomorphisms which are supported by balls in int M).

7. Relative results

In this section we examine the problem of extending the results of the preceding sections to the consideration of proper manifold pairs (M, N). A *locally flat proper manifold pair* (M, N) is a pair of manifolds such that N is a locally flat submanifold of M, properly imbedded as a closed subset. In general when considering an imbedding $h: U \to M$ of a subset U of M and a deformation φ of h, one would like φ to have the property that if h is invariant on N $\big($i.e., $h(U \cap N) \subset N\big)$, then $\varphi(h, t)$ is invariant on N for all t $\big($and likewise if $h \mid U \cap N = 1$, then $\varphi(h, t) \mid U \cap N = 1$ for all $t\big)$. We indicate in what follows how it is possible to obtain this property by strengthening the proofs of the lemma and the main theorem.

If $q < n$, we regard $R^q = R^q \times 0$ as a subspace of R^n in the usual manner. This induces natural inclusions $B^q \subset B^n$ and $T^q \subset T^n$. If A and B are subsets of a space X and if $h: A \to X$ is a map, we say that h is *invariant* [respectively, the *identity*] on B if $h(A \cap B) \subset B$ $[h \mid A \cap B = 1]$.

Remark 7.1. In the statement of Lemma 4.1, we can furthermore require that the deformation ψ have the following property. If an imbedding

$$h \in Q \subset I(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$$

is invariant [identity] on $B^k \times 4B^q$ for any q, $0 \leqq q \leqq n - 1$, then $\psi(h, t)$ is invariant [identity] on $B^k \times 4B^q$ for all t.

PROOF. In order to obtain this property we require that the immersion $\alpha: B^k \times (T^n - D^n) \to B^k \times \operatorname{int} 3B^n$ in the original proof have an additional property. Let the n-ball $4D^n \subset T^n - 2B^n$ be chosen in such a manner that $4D^n \cap T^q = 4D^q$, a q-ball, for each q, $1 \leqq q \leqq n - 1$ $\big($for example, let $4D^n = e([3, 5] \times [-2, 2]^{n-1})$, where $e: R^n \to T^n$ is the covering map of § 2$\big)$. Then the original immersion $\alpha_0: T^n - D^n \to \operatorname{int} 3B^n$ can be chosen so that $\alpha_0 \mid T^q - D^q$ is an immersion of $T^q - D^q$ into $\operatorname{int} 3B^q$ for each q. This can be accomplished by constructing α_0 inductively (on n), using [6, Th. 4.7] coupled with [5, Th. 5.7]. Let α denote the product immersion $1 \times \alpha_0: B^k \times (T^n - D^n) \to B^k \times \operatorname{int} 3B^n$. The proof of the lemma now proceeds as before, taking care to see that the various deformations of the proof preserve the desired invariance properties. The initial deformation ψ_0 will do so if one chooses a natural collar for $\partial B^k \times R^n$ in $B^k \times R^n$ when applying Proposition 3.2. The canonical Schoenflies theorem, which is used to extend $h_2 \mid$ to h_3, preserves the invariance properties of h_2 because the shrinkings and expansions used in the proof of the theorem are natural. Thus, if h is invariant [identity] on $B^k \times 4B^q$, then h_3 will be invariant [identity] on $B^k \times T^q$. The remaining portion of the proof carries through without further modification.

Remark 7.2. Suppose (M, N) is a locally flat proper manifold pair. Then the deformation φ in the statement of Theorem 5.1 can be assumed to have the additional property that if an imbedding $h \in P \subset I(U; M)$ is invariant [identity] on $U \cap N$, then $\varphi(h, t)$ is invariant [identity] on $U \cap N$ for all t.

PROOF. We first indicate how *case* 1 of the proof of the theorem can be modified so that the deformation φ has this additional property. The idea is to initially deform the imbeddings to be the identity on $U \cap W_N$ where W_N is some neighborhood of $(N \cap C) \cup D$. The deformation can then be completed by applying the original *case* 1 of the theorem.

Let $p = \dim N$, and let $\{(W_i, h_i) \mid 1 \leqq i \leqq r\}$ be a finite collection of coordinate neighborhoods of M which lie in U and cover $N \cap \overline{C - D}$, such that $h_i: (W_i, W_i \cap N) \to (R^m, R^p)$ is a homeomorphism of pairs. Let $C_1, \cdots, C_r$ be a collection of compact subsets of $N \cap \overline{C - D}$ which cover $N \cap \overline{C - D}$, such that $C_i \subset W_i$ for each i. We proceed now as in the original proof, using the same induction argument. The definition of D_i is the same, and the induction hypothesis carries the additional assumption that if h is invariant [identity] on N, then $\varphi_i(h, t)$ is invariant [identity] on N. At the $(i+1)^{\text{st}}$ step, the construction of the handlebody pair (K, L) is modified as follows. First, construct a handlebody pair (K_N, L_N) in $W_{i+1} \cap N = R^p$ satisfying properties 1 through 4, with m replaced by p (so that in particular the handles have dimension p). If $\varepsilon > 0$ is chosen small enough and if (K, L) is defined to be $(K_N, L_N) \times \varepsilon B^{m-p}$, then (K, L) is a handlebody pair in W_{i+1} satisfying properties 1 through 4. The handles of (K, L) will have index $\leqq p$ and the imbedding μ of property 4 will in addition be a map of pairs such that $\mu: (B^k \times R^n, B^k \times R^{p-k}) \to (R^m, R^p)$ and $\big(\mu(B^k \times B^n), \mu(B^k \times B^{p-k})\big) = (A, A \cap N)$. The subinduction argument now proceeds as before, carrying along in the induction the additional invariance condition. At the completion of the induction arguments, one has a neighborhood P of the inclusion η in $I(U, U \cap V; M)$ and a deformation $\varphi: P \times I \to I(U, U \cap D; M)$ of P into $\mathrm{I}(U, U \cap W_N; M)$ such that φ satisfies the invariance property, where W_N is some neighborhood of $(N \cap C) \cup D$. The deformation can now be completed either by applying the original *case* 1 to appropriately chosen subsets of $C \cup D$ and U, or by continuing the above proof in the manner of the original proof, being careful to carry out subsequent deformations away from N.

The proof of *case* 2, when $\overline{C - D} \cap \partial M \neq \varnothing$, is just the same as the original proof except for one additional observation. This is that there exists a collar $\sigma: \partial M \times [0, 1] \to M$ for M such that $\sigma \mid \partial N \times [0, 1]$ is a collar for N, i.e., $\sigma\big(\partial N \times [0, 1]\big) \subset N$. This relative version of the collaring theorem is an easy extension of Brown's original proof [2]. It is only necessary to start with local collars for ∂M in M with the property that they restrict to local collars for ∂N in N. If such a relative collar is used in the proof of Proposition 3.2 and in the original proof of *case* 2, then the deformation φ defined there will have the desired invariance properties. This completes the proof of Remark 7.2.

As noted already, the corollaries admit natural extensions to the relative case. For example, considering Corollary 1.1, let (M, N) be a proper manifold pair and let $\mathcal{H}(M, N)$ [respectively $\mathcal{H}_1(M, N)$] denote the subgroup of homeomorphisms of $\mathcal{H}(M)$ which are invariant [identity] on N. Then we have

COROLLARY 7.3. *Let (M, N) be a compact locally flat proper manifold pair. Then the homeomorphism group $\mathcal{H}(M)$ of M is locally contractible in such a manner that the contractions take $\mathcal{H}(M, N)$ into itself and $\mathcal{H}_1(M, N)$ into itself. Thus, in particular, $\mathcal{H}(M, N)$ and $\mathcal{H}_1(M, N)$ are locally contractible.*

We turn now to the proof of Corollary 1.4.

PROOF OF COROLLARY 1.4. As in the proof of Corollary 1.2, it suffices to show that the isotopy $h_t \colon N \to M$ can be covered locally with respect to t, that is, for each $t_0 \in I$ there is a neighborhood $N(t_0)$ of t_0 in I and an isotopy $H_t \colon M \to M$, $t \in N(t_0)$, such that $h_t = H_t h_{t_0}$ for $t \in N(t_0)$. It is enough to consider the case $t_0 = 0$. We can assume without loss of generality that (M, N) is a locally flat proper manifold pair, with N compact, and that $h_0 = 1$.

It follows from the definition of local flatness of h_t and the compactness of N that there is a finite collection $\{U_i \mid 1 \leqq i \leqq r\}$ of open subsets of M which cover N and an $\varepsilon > 0$ such that for each i, the isotopy $h_t \mid U_i \cap N$, $t \in [0, \varepsilon]$, admits a proper extension to U_i, say $h_{i,t} \colon U_i \to M$, $t \in [0, \varepsilon]$, such that $h_{i,0} = 1$. The idea of the proof is to use Remark 7.2 above to show that if $\varepsilon_0 < \varepsilon$ is chosen small enough, then $h_t \colon N \to M$, $t \in [0, \varepsilon_0]$, admits a proper extension to some neighborhood U of N. It then follows from Corollary 1.2 that $h_t \colon N \to M$, $t \in [0, \varepsilon_0]$, can be covered by an isotopy of M.

In what follows it will always be assumed that the various isotopies are defined for t in the interval $[0, \varepsilon]$, and that ε is to be replaced by a smaller ε whenever it becomes necessary. Let $C_1, \cdots, C_r$ be a collection of closed subsets of N which cover N such that $C_i \subset U_i$ for each i. Let $D_i = \bigcup_{1 \leqq j \leqq i} C_j$, and assume inductively that there is a neighborhood V_i of D_i in M such that $h_t \mid V_i \cap N$ admits a proper extension to V_i, say $g_t \colon V_i \to M$, such that $g_0 = 1$. We show how this hypothesis can be extended to hold for some neighborhood V_{i+1} of D_{i+1}, for sufficiently small values of t.

The basic idea is to replace the extension $h_{i+1,t} \colon U_{i+1} \to M$ by one which agrees with $g_t\colon V_i \to M$ on some neighborhood of $D_i \cap C_{i+1}$. Let U_0 and U be compact neighborhoods of $D_i \cap C_{i+1}$ such that $U_0 \subset \operatorname{int} U$ and $U \subset \operatorname{int}(V_i \cap U_{i+1})$, and let ε be small enough so that $h_{i+1,t}(U) \subset g_t(V_i \cap U_{i+1})$ for all $t \in [0, \varepsilon]$. Consider $g_t^{-1}h_{i+1,t} \mid U$, which is 1 on $U \cap N$ and 1 when $t = 0$. If we let

$$P = \{(g_t^{-1}h_{i+1,t} \mid U) \mid t \in [0, \varepsilon]\} ,$$

then it follows from Remark 7.2 that if ε is sufficiently small, there is a deformation $\varphi\colon P \times I \to I(U; M)$ of P into $I(U, U_0; M)$ modulo $\mathrm{fr}_M U$ such that $\varphi(h, t) \mid U \cap N = 1$ for all $h \in P$ and $t \in I$. Thus we can define an isotopy $f_t\colon U_{i+1} \to M$ by

$$f_t = \begin{cases} h_{i+1,t} & \text{on } U_{i+1} - U \\ g_t\varphi(g_t^{-1}h_{i+1,t}, 1) & \text{on } U . \end{cases}$$

Then f_t is an extension of $h_t \mid U_{i+1} \cap N$ and f_t agrees with g_t on U_0.

It remains to use f_t to extend the germ of the isotopy g_t over some neighborhood V_{i+1} of D_{i+1}. Let W_1, W_2 be compact neighborhoods of D_i, C_{i+1} in V_i, U_{i+1} respectively such that $W_1 \cap W_2 \subset \operatorname{int} U_0$. Let $V_{i+1} = W_1 \cup W_2$, and extend the isotopy $g_t \mid W_1\colon W_1 \to M$ to V_{i+1} by letting $g_t \mid W_2 = f_t \mid W_2$. If ε is sufficiently small, then $g_t\colon V_{i+1} \to M$, $t \in [0, \varepsilon]$, is a proper isotopy which extends $h_t \mid V_{i+1} \cap N$. This completes the proof.

8. Alternative proof of Lemma 4.1

The first half of the proof of Lemma 4.1 was concerned with taking a proper imbedding $h\colon B^k \times 4B^n \to B^k \times R^n$ which was close to the identity and producing a homeomorphism $h_3\colon B^k \times T^n \to B^k \times T^n$ which agreed with h on $B^k \times 2B^n$. The principal tools used in this construction were the immersion α of $B^k \times (T^n - D^n)$ into $B^k \times \operatorname{int} 3B^n$ and the canonical Schoenflies theorem. We give below an alternative proof which does not use either of these devices. The following lemma is proved in sufficient generality so that the relative version of Lemma 4.1 (see Remark 7.1) also follows from it.

LEMMA 8.1. *Let $h\colon B^k \times 4B^n \to B^k \times R^n$ be a proper imbedding. If h is sufficiently close to the identity, then there is a homeomorphism $\hat{h}\colon B^k \times T^n \to B^k \times T^n$ with the following properties*

(1) $\hat{h} \mid B^k \times 2B^n = h \mid B^k \times 2B^n$,

(2) *$\hat{h}$ depends continuously on h and if $h = 1$, then $\hat{h} = 1$,*

(3) *if $h \mid \partial B^k \times 4B^n = 1$, then $\hat{h} \mid \partial B^k \times T^n = 1$, and*

(4) *if h is invariant [identity] on $B^k \times R^q$ for any q, $0 \leqq q \leqq n - 1$, then $\hat{h}$ is invariant [identity] on $B^k \times T^q$ (see § 7 for definitions).*

PROOF. The lemma is proved by using an ε-version of the following theorem about the factorization of maps. If $\delta > 0$, let $e_\delta\colon R^1 \to S^1$ be the covering projection defined by $e_\delta(x) = e(\delta x)$.

THEOREM (M. Brown, 1964, unpublished; also [14, p. 535]). *Let X and Y be compact Hausdorff spaces and let $h\colon X \times R \to Y \times R$ be a homeomorphism. Then there exists a $\delta > 0$ and a homeomorphism $g\colon X \times S^1 \to Y \times S^1$ such that the following diagram commutes,*

$$\begin{array}{ccc} X \times [-1, 1] & \xrightarrow{h|} & Y \times R \\ \Big\downarrow{\scriptstyle 1_X \times e_\delta} & & \Big\downarrow{\scriptstyle 1_Y \times e_\delta} \\ X \times S^1 & \xrightarrow{g} & Y \times S^1 . \end{array}$$

The proof of this theorem, except for some connectedness arguments, is implicit in what follows.

In order to avoid excess notation, we will prove only the case $k = 0$. For general k, the proof is exactly the same except that everything is multiplied by B^k.

Let $\alpha_n\colon (-6, 6) \times T^{n-1} \to R^n$ be an imbedding such that $\alpha_n\big((-6, 6) \times T^q\big) \subset R^{q+1}$ for each q, $0 \leqq q \leqq n - 1$. Such an imbedding can be constructed inductively by starting with $n = 1$ and, in general, by obtaining α_{n+1} from α_n by regarding R^n as the subset $(0, \infty) \times R^{n-1} \times 0$ of R^{n+1} and by rotating R^n about the axis $0 \times R^{n-1} \times 0$ in R^{n+1}. If care is used during the construction and if α_n is adjusted when completed, then we can further assume that $\alpha_n \,|\, 2B^n = 1$ and

$$\alpha_n\big((-6, 6) \times T^{n-1}\big) \subset \operatorname{int} 4B^n .$$

Let α denote this final imbedding α_n.

Let $h\colon 4B^n \to R^n$ be an imbedding. If h is sufficiently close to the identity, then $h\alpha\big([-4, 5] \times T^{n-1}\big) \subset \alpha\big((-6, 6) \times T^{n-1}\big)$ and therefore we can define an imbedding

$$h_1 = \alpha^{-1}h\alpha\colon [-4, 5] \times T^{n-1} \to (-6, 6) \times T^{n-1} .$$

At this point we make use of the idea of the theorem mentioned above.

Let $a_0 = -3\ 2/3$, $b_0 = -3\ 1/3$, $a_1 = 4\ 1/3$, and $b_1 = 4\ 2/3$. Define a piecewise linear homeomorphism $\omega_0\colon R \to R$ by $\omega_0(t) = t$ for $t \geqq b_1$, $\omega_0(t) = t + 8$ for $t \leqq a_0$, and let ω_0 map the segment $[a_0, b_1]$ linearly onto the segment $[a_1, b_1]$. Let λ, ω and $\bar{\omega}$ be piecewise linear homeomorphisms of $R \times T^{n-1}$ onto itself given by $\lambda(t, x) = (t - 8, x)$, $\omega = \omega_0 \times 1_{T^{n-1}}$, and $\bar{\omega} = \lambda\omega^{-1}\lambda^{-1}$.

If the imbedding $h_1\colon [-4, 5] \times T^{n-1} \to R \times T^{n-1}$ is sufficiently close to the identity, then we can define an imbedding $h_2\colon [-4, 4] \times T^{n-1} \to R \times T^{n-1}$

by letting $h_2 = \bar{\rho}\bar{\omega}h_1 \mid [-4, 4] \times T^{n-1}$, where $\bar{\rho}\colon D_{\bar{\rho}} \to R \times T^{n-1}$ is an imbedding with domain $D_{\bar{\rho}} = \bar{\omega}h_1([-4, 5] \times T^{n-1})$ and is defined by

$$\bar{\rho} = \begin{cases} (\lambda h_1)\omega(\lambda h_1)^{-1} & \text{on } D_{\bar{\rho}} \cap \lambda h_1([-4, 5] \times T^{n-1}) \\ 1 & \text{elsewhere on } D_{\bar{\rho}} \end{cases}$$

Then h_2 depends continuously on h_1 and $h_2 = 1$ if $h_1 = 1$ (see the diagram below).

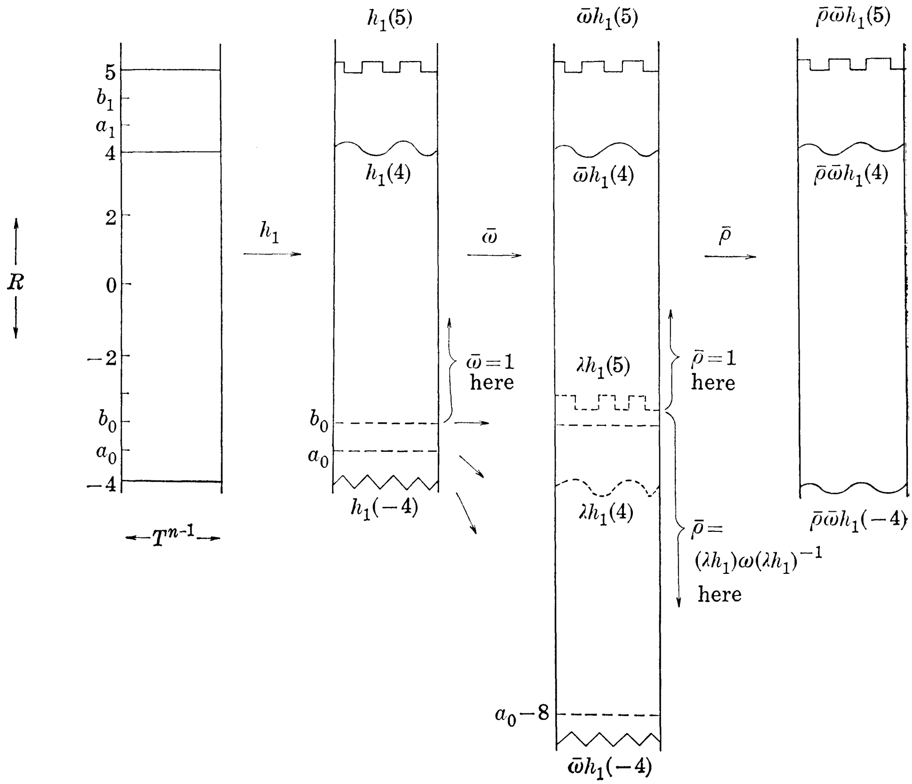

FIGURE 3

By definition, $h_2\lambda = \lambda h_2$ on $\{4\} \times T^{n-1}$. Thus $h_3 = (e \times 1)h_2(e \times 1)^{-1}\colon T^n \to T^n$ is a well defined map which makes the diagram below commute. Furthermore, h_3 is $1 - 1$, and therefore a homeomorphism, if h_2 is close enough to the identity.

$$\begin{array}{ccc} [-4, 4] \times T^{n-1} & \xrightarrow{h_2} & R \times T^{n-1} \\ \Big\downarrow{\scriptstyle e\times 1} & & \Big\downarrow{\scriptstyle e\times 1} \\ S^1 \times T^{n-1} & \xrightarrow{h_3} & S^1 \times T^{n-1} \end{array}$$

Thus h_3 is the desired homeomorphism $\hat{h}$ of the lemma. In general, the invariance properties of h are preserved by h_3 because the stretching and shrinking homeomorphisms are invariant on the appropriate subspaces. This completes the proof.

Remark. The above lemma can be strengthened so that the homeomorphism $\hat{h}: B^k \times T^n \to B^k \times T^n$ actually inherits stronger invariance properties from h than those given by property 4 of the lemma. For example, property 4 can be replaced by

(4′) *if H is a subspace of $R^k \times R^n$ generated by a subcollection of the standard basis vectors for $R^k \times R^n$ and if h is invariant* [*or the identity*] *on H, then $\hat{h}$ is invariant* [*identity*] *on $B_H \times T_H$, where $B_H \times T_H = e(H \cap B^k \times R^n)$.*

Note that this property can be reflected in the deformation ψ defined in Lemma 4.1. Such a version of the lemma would be useful, for example, if one were considering homeomorphisms of manifold n-ads $(M; N_1, \cdots, N_n)$ where each N_i is a proper locally flat submanifold of M and the various combinations of the N_i's intersect nicely (transversally, for example).

To prove this version of the lemma, one uses n applications of the factorization idea used in the proof instead of one application. Thus, starting with the given imbedding $h: B^k \times 4B^n \to B^k \times R^n$, one produces a sequence of proper imbeddings $h = h_0, h_1, \cdots, h_n$, where

$$h_j: B^k \times T^j \times (4 - j/n)B^{n-j} \to B^k \times T^j \times R^{n-j}$$

is such that h_j and h_{j+1} agree on $B^k \times T^j \times 2B^{n-j}$. Each imbedding is produced directly from the preceding one by "identifying" $B^k \times T^j \times \{3\} \times 4B^{n-j-1}$ with $B^k \times T^j \times \{-3\} \times 4B^{n-j-1}$ to produce $B^k \times T^{j+1} \times 4B^{n-j-1}$. The final imbedding h_n is the desired homeomorphism $\hat{h}$.

UNIVERSITY OF MICHIGAN
UNIVERSITY OF CALIFORNIA, LOS ANGELES
INSTITUTE FOR ADVANCED STUDY

REFERENCES

[1] M. BROWN, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

[2] ———, *Locally flat imbeddings of topological manifolds*, Ann. of Math. **75** (1962), 331–341.

[3] A. V. CERNAVSKII, *Local contractibility of the homeomorphism group of a manifold*, Soviet Math. Dokl. **9** (1968), 1171–1174.

[4] G. M. FISHER, *On the group of all homeomorphisms of a manifold*, Trans. Amer. Math. Soc. **97** (1960), 193–212.

[5] M. W. HIRSCH, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276.

[6] ———, *On imbedding differentiable manifolds in euclidean space*, Ann. of Math. **73** (1961), 566–571.

[7] J. F. P. HUDSON, *Extending piecewise linear isotopies*, Proc. Lond. Math. Soc. (3) **16** (1966), 651-668.

[8] ———, Piecewise Linear Topology, W. A. Benjamin, Inc., New York, 1969.

[9] R. C. KIRBY, Stable homeomorphisms and the annulus conjecture, Ann. of Math. **89** (1969), 575-582.

[10] J. M. KISTER, *Microbundles are fibre bundles*, Ann. of Math. **80** (1964), 190-199.

[11] N. H. KUIPER and R. K. LASHOF, *Microbundles and bundles*: I. Elementary theory, Invent. Math. **1** (1966), 1-17.

[12] J. A. LEES, *Immersions and surgeries of topological manifolds*, Bull. Amer. Math. Soc. **75** (1969) 529-534.

[13] D. BARDEN, *A quick immersion*, to appear.

[14] L. C. SIEBENMANN, *Pseudo-annuli and invertible cobordisms*, Arch. Math. **19** (1968), 528-535.

[15] D. GAULD, Mersions of topological manifolds, Ph. D. thesis, Univ. of Cal., Los Angeles, 1969.

[16] W. HUEBSCH and M. MORSE, *The dependence of the Schoenflies extension on an accessory parameter*, J. Analyse Math. **8** (1960/61), 209-221.

(Received July 15, 1969)